CONTENTS

1. INTRODUCTION

Fundametally, metric spaces, curves, surfaces and other such spaces are just sets with certain structures. They all have properties unique to that kind of space that arise from the structure. Topology is a way of combining all of these kinds of spaces under a general definition. *Algebraic* topology is the concept of assigning algebraic objects, such as groups, to these spaces to find out more about them.

Topology is sometimes called "rubber sheet geometry," because regardless of how much a space is stretched and deformed, as long as it is not torn or pierced, it retains the same topological properties; the surface of a football is topologically equivalent to that of a rugby ball, and the surface of a doughnut is topologically equivalent to that of a teacup. When two topological spaces share the same topological properties, they are known as homeomorphic. It is one of the algebraic topologist's aims to be able to work out whether two spaces are homeomorphic or not.

Proving whether two spaces are homeomorphic is a case of finding a homeomorphism between them. Proving that two spaces are not homeomorphic is more difficult. In 1894, Jules Henri Poincaré introduced the *first homotopy group*, or fundamental group. This is a group assigned to a space determined by the different types of closed paths, or loops, that the space has. If two spaces are homeomorphic then they have isomorphic fundamental group, and so two spaces can be shown to be not homeomorphic by proving that they do not have isomorphic fundamental groups. Furthermore, because the spaces are being described using groups, topologists have all the tools of group theory to work with.

In this project we will calculate the fundamental groups of various types of spaces, most notably a type of surface called the compact orientable surface. In the process we will see that any compact orientable surface is homeomorphic to one of a set of surfaces that should be familiar: the sphere, the torus, the double torus, and so on. We will also look at an application of algebraic topology in another area of mathematics altogether: we will use the fundamental group of the circle to solve the fundamental theorem of algebra. We will start, though, with some basic definitions.

2. TOPOLOGICAL SPACES

Definition. A **topological space** $\{X, \mathcal{T}\}$ consists of a non-empty set X together with a fixed collection $\mathcal{T}$ of subsets of X satisfying:

1. X and $\emptyset$ are in $\mathcal{T}$;
2. any arbitrary union of sets is $\mathcal{T}$ is in $\mathcal{T}$;
3. any finite intesection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$\mathcal{T}$ is called a **topology** for X, and the members of $\mathcal{T}$ are called the **open sets** of the space.

We normally denote the topological space $\{X, \mathcal{T}\}$ by just X. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies for a space X and $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$, and that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$. If X is any set, the collection of all subsets of X is a topology on X, called the **discrete topology**, and $\{X, \emptyset\}$ is a topology on X called the **indiscrete** or **trivial topology**.

The definition of a topological space also gives us a definition for an open set in a topological space; as there is in general no notion of distance in such a space, we cannot define it using open balls as in metric spaces. We define continuity of functions the same way as we do in metric spaces. Letting X and Y be topological spaces, a function $f : X \longrightarrow Y$ is **continuous** if for each open set U of Y, the set $f^{-1}(U)$ is an open set of X.

A subset V of a topolgical space X is called **closed** if $X - V$ is open. Both $\emptyset$ and X are closed, as are finite unions and arbitrary intersections of closed sets. The definition of a continuous function can be rewritten as above but with the word "closed" substituted for the word "open".

As the definition for a topolgical space is so general, it is unsurprising that we can compare and link different topological spaces. In the same way that we link algebraic objects like groups with the idea of an isomorphism, we can link topological spaces using the idea of a homeomorphism.

Definition. Let X and Y be two topological spaces, and let $f : X \longrightarrow Y$ be a bijection. If f and f^{-1} are both continuous, then f is a **homeomorphism** between X and Y. We say that X and Y are **homeomorphic**, and we write $X \cong Y$.

Another way of thinking about this definition is by saying that a homeomorphism is a bijective correspondence $f : X \longrightarrow Y$ such that for a subset U of X, the set $f(U)$ is open if and only if U is open. Two topological spaces which are homeomorphic share the same topolgical structure.

The Basis of a Topology

It is sometimes easier to define a topology for a space X in terms of a basis; a collection $\mathcal{B}$ of subsets of X satisfying:

1. for each $x \in X$ there is at least one basis element B containing x;
2. if $x \in B_1 \cap B_2$, then there exists B_3 such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

The open sets of the topology $\mathcal{T}$ generated by $\mathcal{B}$ are all the sets U such that for each element x in U there exists a basis element $B \in \mathcal{B}$ with $x \in B$ and $B \subset U$. This consists of all the sets of $\mathcal{B}$ together with all unions of such sets. It is easy to check this is a topology:

(1) The empty set is in $\mathcal{T}$ as it satisfies the conditions for openness vacuously, and X is in $\mathcal{T}$ since for each x in X there is some basis element in X containing x.

(2) Let
$$U = \bigcup_{\alpha \in J} U_\alpha,$$
where U_α is an element of $\mathcal{T}$ for all α in some indexing set J. Given $x \in U$, there is some U_α containing x, and so there exists a basis element B satisfying $x \in B \subset U_\alpha$. Therefore $x \in B \subset U$, and so U is in $\mathcal{T}$.

(3) Let
$$U = U_1 \cap U_2,$$
where U_1 and U_2 are elements of $\mathcal{T}$. Given $x \in U$, there are basis elements B_1 and B_2 satisfying $x \in B_1 \subset U_1$ and $x \in B_2 \subset B_2$, and so there exists a basis element B_3 satisfying $x \in B_3 \subset B_1 \cap B_2$. Therefore $x \in B_3 \subset U$, and so U is in $\mathcal{T}$. It follows by induction that any finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

Therefore $\mathcal{T}$ is a topology for X.

Compact Orientable Surfaces

Some examples of topological space are the surfaces of the sphere, the torus, the double torus, and so on, as shown in Figure 1. These are special cases of n-dimensional constructions called **manifolds**, which we can define in terms of some properties of topological spaces:

Definition. A **separation** of a topological space X is a pair U, V of disjoint non-empty open subsets of X such that $X = U \cup V$. The space X is called **connected** if there does not exist a separation of X.

Definition. A **cover** of a space X is a collection of subsets $\{U_j : j \in J\}$ of X, where J is some indexing set, such that X is in the union of all the sets U_j. It is called an **open cover** if U_j is open for all $j \in J$, and it is called a **finite cover** if J is finite. A topological space X is **compact** if every open cover of X has a finite subcover.

Definition. A topological space X is a **Hausdorff space** if, for every pair of distinct points x and y in X, there exist open sets U_1 containing x and U_2 containing y such that $U_1 \cap U_2 = \emptyset$.

Definition. An **n-dimensional manifold** or **n-manifold** is a Hausdorff space in which each point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. A compact connected 2-manifold is called a **compact surface**. Compact surfaces which do not contain möbius strips are called **compact orientable surfaces**.

Theorem. ***The Classification Theorem of Surfaces.***
If we denote the sphere S^2 by Σ_0, the torus T^2 by Σ_1, the double torus by Σ_2 and the n-holed torus by Σ_n, then any compact orientable surface is homeomorphic to precisely one of the surfaces Σ_k, for a non-negative integer k.

We call k the surface's **genus**. The proof of this theorem is in two parts: to show that any compact orientable surface is homeomorphic to at least one of Σ_k, and to show that two surfaces Σ_k and Σ_l for $k \neq l$ are not homeomorphic. We shall not prove the former part – an outline of the proof can be found in [2] – but we shall prove the latter part later on. The theorem is particularly useful because, as we shall discover later, it is very easy to find the fundamental group of Σ_k for any k.

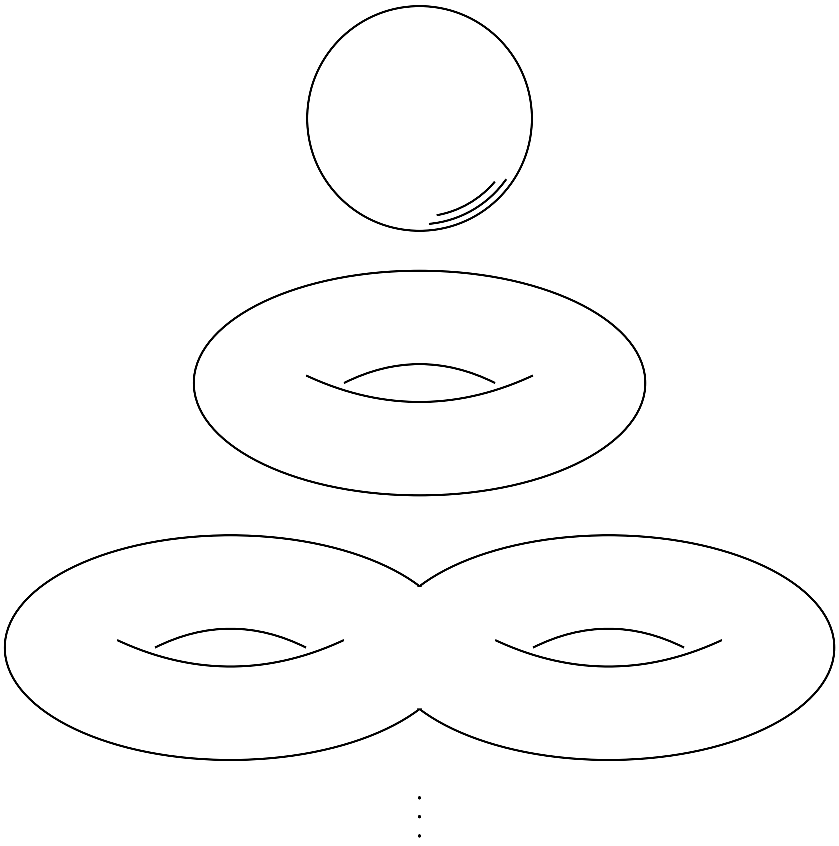


Figure 1

3. Creating New Spaces

There are several ways of constructing topological spaces using what we know about other spaces. This means that we can define a large number of complicated spaces just using simple spaces, such as the plane $\mathbb{R}^2$ for example. The first two of these methods we will look at, the product topology and the subspace topology, are quite intuitive. We will then look at the quotient topology, which will prove very useful when we go on to find the fundamental groups of compact orientable surfaces.

The Product Topology

Let X and Y be topological spaces. Let $\mathcal{B}$ be the collection of sets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y. Then we claim that $\mathcal{B}$ is a basis for a topology on $X \times Y$.

The first condition for $\mathcal{B}$ to be a basis is trivial, since $X \times Y$ is itself a basis element. Now, for basis elements $U_1 \times V_1$ and $U_2 \times V_2$,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

The sets $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, so $(U_1 \times V_1) \cap (U_2 \times V_2)$ is a basis element. Therefore the second condition holds.

The topology on $X \times Y$ generated by this basis is called the **product topology**. An example of a space with the product topology is the torus T^2, which can be created by taking the product of two circles, $S^1 \times S^1$.

The Subspace Topology

If X is a topological space with topology $\mathcal{T}$, and Y is a subset of X, then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is a topolgy on Y, called the **subspace topology**.

It is easy to see that $\mathcal{T}_Y$ is a topology: it contains $\emptyset$ and Y, since $\emptyset = Y \cap \emptyset$ and $Y = Y \cap X$, and it is closed under finite intersections and arbitrary unions, since

$$\bigcap_{\alpha=1}^{n} (U_\alpha \cap Y) = \left(\bigcap_{\alpha=1}^{n} U_\alpha \right) \cap Y$$

for some integer n and

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha \right) \cap Y$$

for some indexing set J.

Note that if Y is open in X, then all open sets in Y are also open sets in X. If $\mathcal{T}$ is defined in terms of a basis, then this lemma, proved in [4], can be used:

Lemma. *If $\mathcal{B}$ is a basis for $\mathcal{T}$ on X, then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$ on Y.*

The Quotient Topology

Imagine taking a length of wire and bending it round until the two ends meet so that you get a circular ring. With one simple movement you have turned one space into a completely different one. If we think of the wire as the closed interval $[0, 1]$ and the ring as S^1, then there is a function $f : [0, 1] \longrightarrow S^1$ which defines this change:

$$f(t) = (\cos 2\pi t, \sin 2\pi t).$$

For a slightly more complicated example, imagine taking a rectangular piece of card and identifying two opposite edges. You get a cylindar. If you now identify the other two edges you end up with a torus. Again we can define this creation of a new space with a function. Let

$$X = \{(u, v) \in \mathbb{R}^2 : 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$$

be a unit square and

$$Y = \{(x, y, z) \in \mathbb{R}^3 : ((x^2 + y^2)^{\frac{1}{2}} - 2)^2 + z^2 = 1\}$$

be a torus. Then we can define $f : X \longrightarrow Y$ by

$$f(u, v) = ((\cos 2\pi u + 2)\cos 2\pi v, (\cos 2\pi u + 2)\sin 2\pi v, \sin 2\pi u).$$

It is easy to imagine that we can define a topology for a new space Y in terms of a topology for an original space X and a surjective function $f : X \longrightarrow Y$, and indeed we can.

Definition. Let X and Y be topolgical spaces and let $f : X \longrightarrow Y$ be a surjective map. The map f is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X. There is exactly one topology on Y relative to which f is a quotient map, and it is called the **quotient topology** on Y determined by f.

To check this fits the definition of a topology, we can see that $\emptyset$ and Y are open since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, and for open sets U_α of Y, we can see that

$$\bigcap_{\alpha=1}^{n} U_\alpha$$

is open for some integer n since

$$f^{-1}(\bigcap_{\alpha=1}^{n} U_\alpha) = \bigcap_{\alpha=1}^{n} f^{-1}(U_\alpha)$$

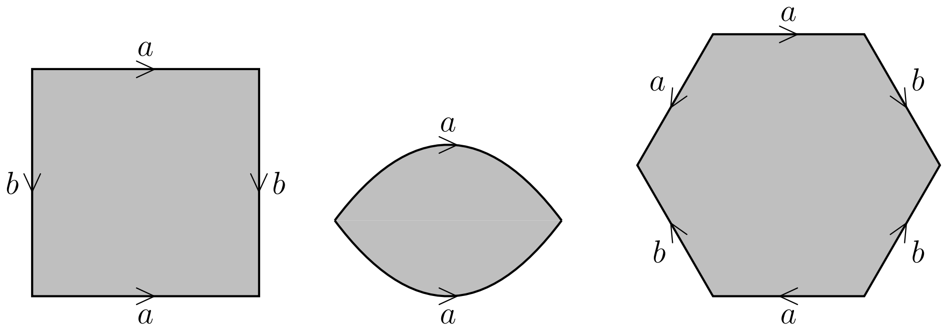
and

$$\bigcup_{\alpha \in J} U_\alpha$$

is open for some indexing set J since

$$f^{-1}(\bigcup_{\alpha \in J} U_\alpha) = \bigcap_{\alpha \in J} f^{-1}(U_\alpha).$$

We draw the fact that we wish to identify sides of a space with labeled arrows as in Figure 2, which also shows some more examples. Notice that we do not need to confine ourselves to spaces that we can visualise with pieces of wire and card.



Torus Sphere

FIGURE 2

It is not obvious that the torus T^2 created in this way is homeomorphic to the product space $S^1 \times S^1$, especially as, if we call the torus with the quotient topology X and the torus with the product topology Y, then X is a subspace of $\mathbb{R}^3$ and Y is a subspace of $\mathbb{R}^4$. The function $f : X \longrightarrow Y$ given by

$$f(u, v) = ((\cos 2\pi u, \sin 2\pi u), (\cos 2\pi v, \sin 2\pi v))$$

is a homeomorphism; we will leave it to the reader to check this.

We can generalise the definition of the quotient topology as follows:

Let Y be a set, $\{X_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of topological spaces, and $\{f_\lambda : X_\lambda \longrightarrow Y : \lambda \in \Lambda\}$ be an arbitrary family of maps. Then a set $U \subset Y$ is open if and only if $f_\lambda^{-1}(U)$ is open in X_λ for all $\lambda \in \Lambda$.

Using this generalisation we can see that it is possible to create a new topological space not only by deforming existing ones but by 'sticking' two or more spaces together. If each of the above maps $f_\lambda : X_\lambda \longrightarrow Y$ is one-one and the images $f_\lambda(X_\lambda)$ are pairwise disjoint and cover Y, then we say that Y is the **topological sum** of the collection of spaces X_λ.

Example. Consider the spaces in Figure 3. Identifying the sides labeled a, b, c and d gives us two tori, each with an open disc removed. Taking the topological sum of the tori by identifying the sides labeled e gives us a double torus.

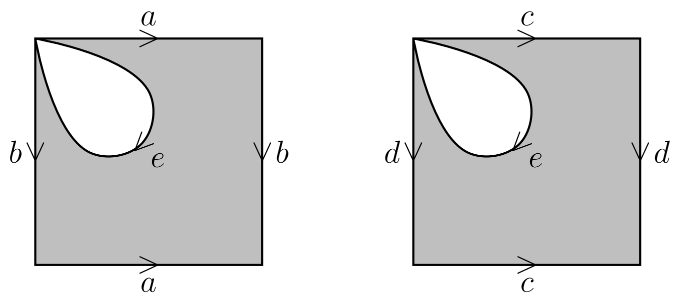


FIGURE 3

However, we can actually define the double torus in the quotient topology using a single polygon. Opening up the squares to give pentagons, and identifing the sides e gives an octagon as in Figure 4.

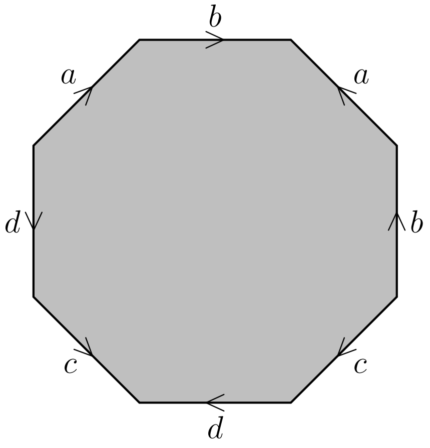


FIGURE 4

Similar constructions show that an n-holed torus can be created from a $4n$-sided polygon, with the side identification labels being of the form

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}.$$

4. HOMOTOPY

Homotopy is the concept of continuous transformations from one function to another, and is main part of the construction of the fundamental group.

Definition. Consider two continuous maps f_1 and f_2 from the space X to the space Y such that, letting I equal the interval $[0,1]$, there is a continuous map $F : X \times I \longrightarrow Y$ with $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$, for all x in X. We can think of F as a deformation of the map f_1 into the map f_2 as time goes from 0 to 1. If such a map exists, then we say that f_1 is **homotopic** to f_2, written $f_1 \simeq f_2$, and that F is a **homotopy** between them.

Example. Any two maps f_1 and f_2 from a space X to $\mathbb{R}^2$ are homotopic. The homotopy between them given by the equation

$$F(x,t) = (1-t)f_1(x) + tf_2(x)$$

is called the straight line homotopy.

We can show that homotopy is an equivalence relation as follows.

Reflexive: It is trivial that $f \simeq f$ for all f. The required homotopy is the map $F(x,t) = f(x)$ for all t.
Symmetric: Let $f_1 \simeq f_2$, and let F be a homotopy between them. Then $G(x,t) = F(x,1-t)$ is a homotopy between f_2 and f_1, and so $f_2 \simeq f_1$.
Transitive: Suppose $f_1 \simeq f_2$ and $f_2 \simeq f_3$, and let F_1 and F_2 be a homotopy between f_1 and f_2 and a homotopy between f_2 and f_3 respectively. Define $G : X \times I \longrightarrow Y$ by

$$G(x,t) = \begin{cases} F_1(x,2t) & \text{if } t \in [0,\frac{1}{2}] \\ F_2(x,2t-1) & \text{if } t \in [\frac{1}{2},1]. \end{cases}$$

Since, when $t = \frac{1}{2}$,

$$F_1(x,2t) = F_1(x,1) = f_2(x) = F_2(x,0) = F_2(x,2t-1),$$

the map G is well defined. It is continuous by the following lemma:

Lemma. ***The Pasting Lemma.***
Let $X = A \cup B$ where A and B are closed in X. Let $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \longrightarrow Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } s \in A \\ g(x) & \text{if } s \in B \end{cases}$$

is continuous.

The proof of this lemma is in [4]. This shows that G is a homotopy between f_1 and f_3, and so $f_1 \simeq f_3$.

Definition. A continuous map $f : X \longrightarrow Y$ is called a **homotopy equivalence** if there is a continuous map $g : Y \longrightarrow X$ such that $g \circ f$ is homotopic to the identity map e_X of X and $f \circ g$ is homotopic to the identity map e_Y of Y. The map g is called a **homotopy inverse** for f. If there is a homotopy equivalence between two spaces, those spaces are said to be **homotopically equivalent**, or of the same **homotopy type**.

Homeomorphic spaces are of the same homotopy type, but the converse is not true.

Homotopy of Paths

Definition. Again let $I = [0, 1]$. If $f : I \longrightarrow X$ is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$, we say f is a **path** in X from x_0 to x_1. Two such paths f_1 and f_2 are called **path-homotopic** if there is a continuous map $F : I \times I \longrightarrow X$ such that

$$F(s, 0) = f_1(s), \;\; F(s, 1) = f_2(s) \text{ for all } s \in I,$$

$$F(0, t) = x_0, \;\; F(1, t) = x_1 \text{ for all } t \in I;$$

that is, they are homotopic and they keep the same initial and final points throughout the deformation. The map F is called a **path-homotopy** between f_1 and f_2, and we write $f_1 \simeq_p f_2$.

Like homotopy, path-homotopy is an equivalence relation; we leave it to the reader to show this. We denote the path-homotopy equivalence class of a path f by $[f]$.

The path-homotopy classes satisfy properties that look very much like the axioms for a group. Before we show this we need to define the **concatenation** of two paths. If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, then the concatenation $f * g$ of f and g is defined as the path h given by

$$h(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$

Since

$$f(2 \cdot \frac{1}{2}) = f(1) = x_1 = g(0) = g(2 \cdot \frac{1}{2} - 1),$$

the function h is well defined, and it is continuous by the Pasting Lemma.

We can also show that $*$ is well defined on path-homotopy classes: let F be a path-homotopy between maps f_1 and f_2 from x_0 to x_1, and let G be a path-homotopy between maps g_1 and g_2 from x_1 and x_2. Define

$$H(s,t) = \begin{cases} F(2s, t) & \text{if } s \in [0, \frac{1}{2}] \\ G(2s-1, t) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

The function H is a path-homotopy between $f_1 * g_1$ and $f_2 * g_2$. Since $F(1,t) = x_1 = G(0,t)$ for all t, the map H is well-defined, and it is continuous again by the Pasting Lemma. We can therefore define

$$[f] * [g] = [f * g].$$

Theorem. *The path-homotopy classes have the following group-like properties, called the* ***groupoid properties****:*

(1) ***Associativity.***
If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$, and they are equal.

(2) ***Right and left identities.***
Given $x \in X$, let $c_x : I \longrightarrow X$ be the constant path taking all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f] * [c_{x_1}] = [f] \text{ and } [c_{x_0}] * [f] = [f].$$

(3) ***Inverses.***
Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path $\bar{f}(s) = f(1-s)$, called the ***reverse*** *of f. Then*

$$[f] * [\bar{f}] = [c_{x_0}] \text{ and } [\bar{f}] * [f] = [c_{x_1}].$$

Proof.

(1) We need to show that $f * (g * h) \simeq_p (f * g) * h$. The map $f * (g * h)$ takes the point s to $f(2s)$ for s in the interval $[0, \frac{1}{2}]$, takes s to $g(4s - 2)$ for s in $[\frac{1}{2}, \frac{3}{4}]$, and takes s to $h(4s - 3)$ for s in $[\frac{3}{4}, 1]$. The map $(f * g) * h$ traces out the same image but at a different rate. It takes s to $f(4s)$ for s in $[0, \frac{1}{4}]$, takes s to $g(4s - 1)$ for s in $[\frac{1}{4}, \frac{1}{2}]$, and takes s to $h(2s - 1)$ for s in $[\frac{1}{2}, 1]$. We can define a path homotopy $F : I \times I \longrightarrow X$ between these maps by

$$F(s) = \begin{cases} f(2s(1+t)) & \text{if } s \in [0, \frac{2-t}{4}] \\ g(4s + t - 2) & \text{if } s \in [\frac{2-t}{4}, \frac{3-t}{4}] \\ h((4-2t)s + 2t - 3) & \text{if } s \in [\frac{3-t}{4}, 1]. \end{cases}$$

We will leave it to the reader to check that F is well defined and that it satisfies the necessary conditions for it to be our required path homotopy.

(2) We need to show that $f * c_{x_1} \simeq_p f$. The map $f * c_{x_1}$ takes the interval $[0, \frac{1}{2}]$ onto the image of f and the interval $[\frac{1}{2}, 1]$ onto the point x_1, and so we can define a path homotopy $G : I \times I \longrightarrow X$ from $f * c_{x_1}$ to f by

$$G(s,t) = \begin{cases} f((2-t)s) & \text{if } s \in [0, \frac{1+t}{2}] \\ x_1 & \text{if } s \in [\frac{1+t}{2}, 1]. \end{cases}$$

Again, we will leave it to the reader to check that G is well defined and is the desired path homotopy. We can similarly prove that $c_{x_0} * f \simeq_p f$.

(3) We finally need to show that $f * \bar{f} \simeq_p c_{x_0}$. The map $f * \bar{f}$ takes the point s to $f(2s)$ for s in the interval $[0, \frac{1}{2}]$, and takes s to $f(2(1-s))$ for s in $[\frac{1}{2}, 1]$. The map c_{x_0} takes the whole of I to $f(0)$. Therefore we can define a path homotopy $H : I \times I \longrightarrow X$ between the maps by

$$H(s,t) = \begin{cases} f(2st) & \text{if } s \in [0, \frac{1}{2}] \\ f(2t(1-s)) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

The reader can check that this is the required well defined path homotopy. A similar argument shows that $\bar{f} * f \simeq_p c_{x_1}$.

□

These properties differ from the axioms for a group in that $[f] * [g]$ is not defined for every pair $[f]$ and $[g]$. The concatenation $[f] * [g]$ only makes sense if $f(1) = g(0)$. However, if we take a base point x_0 in the space X and consider only those paths which are **loops** from x_0 back to x_0, then this condition will always hold. Therefore these paths will form a group, and it is this that is the **fundamental group** of X. We denote the fundamental group of X relative to the base point x_0 by $\pi_1(X, x_0)$.

5. The Fundamental Group

We often talk about the fundamental group of a space without drawing attention to the base point relative to which it has been calculated. The following theorem shows how fundamental groups of a space relative to different base points are related.

Theorem. *Let X be a path connected space, that is, every pair of points in X can be connected by a path in X. Given two points x_0 and x_1 in X, the fundamental group $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Proof. We define a map $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

where α is a path from x_0 to x_1, and $\bar{\alpha}$ is its reverse, as pictured in Figure 5. We can now show that $\hat{\alpha}$ is a group isomorphism:

It is a homomorphism since

$$\begin{aligned}\hat{\alpha}([f]) * \hat{\alpha}([g]) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\ &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= \hat{\alpha}([f] * [g]).\end{aligned}$$

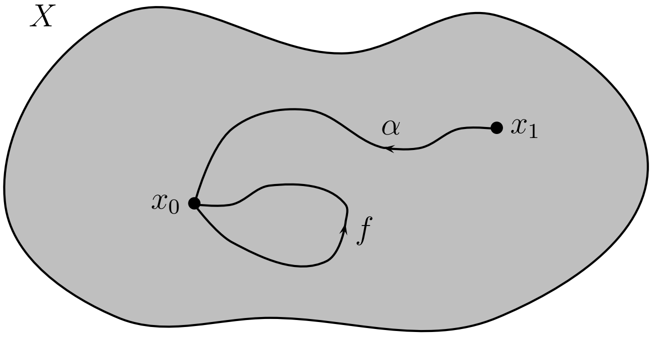


Figure 5

Let $\beta = \bar{\alpha}$. Then

$$\hat{\beta}([f]) = [\bar{\beta}] * [f] * [\beta] = [\alpha] * [f] * [\bar{\alpha}],$$

and so

$$\hat{\alpha}(\hat{\beta}([f])) = [\bar{\alpha}] * ([\alpha] * [f] * [\bar{\alpha}]) * [\alpha] = [f].$$

Similarly, $\hat{\beta}(\hat{\alpha}([f])) = [f]$. This shows that $\hat{\beta}$ is an inverse for $\hat{\alpha}$. Therefore $\hat{\alpha}$ is an isomorphism. ☐

We call a path connected space X for which $\pi_1(X, x_0)$ is the trivial group for some x_0 (and therefore all $x \in X$) **simply connected**. We can denote the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = e$.

Example. We can show that the fundamental group of $\mathbb{R}^n$ is trivial: if $f : I \longrightarrow \mathbb{R}^n$ is a loop in $\mathbb{R}^n$ based at x_0, then the straight line homotopy

$$F(s,t) = tx_0 + (1-t)f(s)$$

is a path-homotopy between f and the constant loop c_{x_0}. More generally, if X is any convex subset of $\mathbb{R}^n$, then the straight line homotopy will still work, as the straight line segment between any two points in X is in X. Therefore, a convex subspace of $\mathbb{R}^n$ is simply connected.

Group Presentation

It is sometimes useful to express the fundamental group of a space using **group presentation**. Consider a set S. A **word** in the elements of S is an expression of the form

$$W = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_k^{\epsilon_k},$$

where $x_1, x_2, \dots, x_k$ are elements of S (allowing repeats) and $\epsilon_i = \pm 1$. We also define the empty word as the word containing no symbols. If a word contains xx^{-1} or $x^{-1}x$ for some $x \in S$, then it can be **reduced** by removing that pair of elements from it.

Using juxtaposition of words as composition, and reducing the resulting words if necessary, it turns out that the set G of all reduced words in the elements of S forms a group, with the empty word as the identity, called the **free group generated by** S. For example, the free group on one generator $\{x\}$ consists of the elements

$$1, x, x^{-1}, x^2, x^{-2}, x^3, x^{-3}, \dots,$$

and is isomorphic to $(\mathbb{Z}, +)$.

Say we now wanted to write down the cyclic group $(\mathbb{Z}_n, +)$ using generators. This is also generated by the element x, but we need to express the fact that

$x^n = 1$. This is called a **relation**, and we can write the group as

$$\langle x \mid x^n = 1 \rangle.$$

Any group can be written in group presentation style as $\langle G \mid R \rangle$, where G is a set of generators and R is a set of relations.

Example. The group $\langle x, y \mid xyx^{-1}y^{-1} = 1 \rangle$ is isomorphic to $(\mathbb{Z}, +) \times (\mathbb{Z}, +)$, more properly written as $\mathbb{Z} \oplus \mathbb{Z}$. The generators x and y represent the elements $(1, 0)$ and $(0, 1)$ respectively, with the empty word representing $(0, 0)$.

The Homomorphism Induced by h

Suppose a function h from a space X to a space Y is continuous, and that $h(x_0) = y_0$, where $x_0 \in X$ and $y_0 \in Y$. We can write this as $h : (X, x_0) \longrightarrow (Y, y_0)$. If $f : I \longrightarrow X$ is a loop in X based at x_0, then $h \circ f : I \longrightarrow Y$ is a loop in Y based at y_0. We can therefore define a homomorphism $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ by

$$h_*([f]) = [h \circ f].$$

This is called the **homomorphism induced by** h. It is well defined since if f_1 and f_2 are path homotopic, and $F : I \times I \longrightarrow X$ is a path homotopy between them, then $h \circ F$ is a path homotopy between the loops $h \circ f_1$ and $h \circ f_2$.

To check h_* is a homomorphism, we need to show that

$$h_*([f] * [g]) = h_*([f]) * h_*([g]).$$

This is easy. By definition,

$$(f * g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2s - 1) & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

and so

$$(h \circ (f * g))(s) = h((f * g)(s)) = \begin{cases} h(f(2s)) & \text{if } s \in [0, \frac{1}{2}] \\ h(g(2s - 1)) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

But this also equals $((h \circ f) * (h \circ g))(s)$. Therefore

$$[h \circ (f * g)] = [(h \circ f) * (h \circ g)] = [(h \circ f)] * [(h \circ g)],$$

and so

$$h_*([f] * [g]) = h_*([f]) * h_*([g]).$$

Theorem. *The induced homomorphism has the following properties, called the* ***functorial properties****.*

(1) *If* $h : (X, x_0) \longrightarrow (Y, y_0)$ *and* $k : (Y, y_0) \longrightarrow (Z, z_0)$*, then* $(k \circ h)_* = k_* \circ h_*$.

(2) *If* $e : (X, x_0) \longrightarrow (X, x_0)$ *is the identity map, then* $e_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$ *is the identity homomorphism.*

Proof. By definition,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f]$$

and

$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)],$$

and so $(k \circ h)_* = k_* \circ h_*$.

Also, $e_*([f]) = [i \circ f] = [f]$. □

From these properties, we can prove a very important result. If $h : (X, x_0) \longrightarrow (Y, y_0)$ is a homeomorphism of X with Y, and $k : (Y, y_0) \longrightarrow (X, x_0)$ is its inverse, then $k_* \circ h_* = (k \circ h)_* = (e_X)_*$, where e_X is the identity map of (X, x_0), and $h_* \circ k_* = (h \circ k)_* = (e_Y)_*$, where e_Y is the identity map of (Y, y_0). Since $(e_X)_*$ and $(e_Y)_*$ are the identity homomorphisms of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ respectively, k_* is the inverse of h_*. Therefore h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

This shows that homeomorphic topological spaces have isomorphic fundamental groups. It is also possible to show that homotopically equivalent spaces have isomorphic fundamental groups, though the proof of this is much more complicated. See [4].

6. Calculating the Fundamental Group

We can now begin calculating some fundamental groups. There are various different methods. The first fundamental group we will find is that of the circle. To do this, we need to introduce the concepts of covering spaces and liftings.

Covering Spaces, Liftings and the Fundamental Group of the Circle

Imagine a continuous surjective map $p : E \longrightarrow B$ such that for an open set U of B, the preimage $p^{-1}(U)$ takes the form of a collection of disjoint open sets V_α in E, and the restriction of p to V_α is a homeomorphism of V_α onto U. The open set U is said to be **evenly covered** by p. An easy to visualise example is shown in Figure 6. Think of $p^{-1}(U)$ as a stack of pancakes, each having the same size and shape as U, with p squashing them all down onto U.

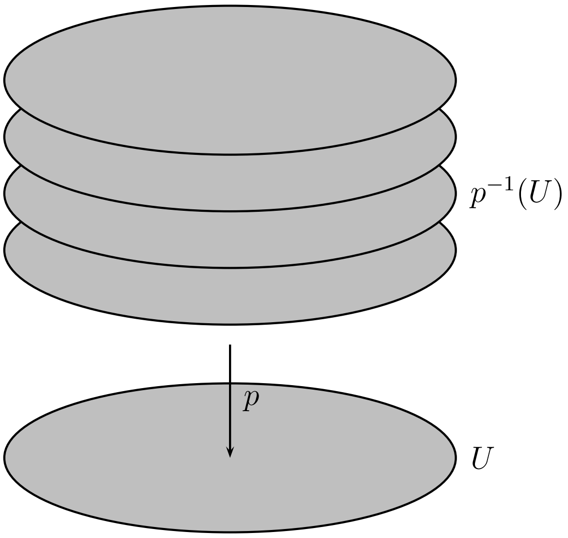


FIGURE 6

If every point in B has a neighbourhood U that is evenly covered by p, then p is called a **covering map**, and B is called a **covering space** of E.

Example. We have seen that we can give S^1 the quotient topology determined by the function $f : [0,1] \longrightarrow S^1$ given by

$$f(x) = (\cos 2\pi x, \sin 2\pi x),$$

so that for every open set U in S^1, the preimage $f^{-1}(U)$ is an open set in $[0,1]$, starting at the point e_0, say. If $p : \mathbb{R} \longrightarrow S^1$ is the extention of f to the whole of the real line, then $p^{-1}(U)$ is a collection of open sets, starting at $e_0 + n$, for all integers n. Therefore U is evenly covered by p, and so p is a covering map. We can think of p as a function which wraps the real line around the circle, mapping each interval [n,n+1] onto S^1.

Now let $p : E \longrightarrow B$ be a map and $f : X \longrightarrow B$ be continuous. A **lifting** of f is a map $\tilde{f} : X \longrightarrow E$ such that $p \circ \tilde{f} = f$, as shown in Figure 7.

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B
\end{tikzcd}$$

Figure 7

The following two lemmas are proved in [4].

Lemma. ***The Path Lifting Lemma.***
Let $p : E \longrightarrow B$ be a covering map, and $p(e_0) = b_0$. Any path $f : I \longrightarrow B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

Example. Let $p : \mathbb{R} \longrightarrow S^1$ be the covering map $p(x) = (\cos 2\pi x, \sin 2\pi x)$.

The path $f : I \longrightarrow S^1$ given by $f(s) = (\cos \pi s, \sin \pi s)$ lifts to $\tilde{f}(s) = \dfrac{s}{2}$.

The path $g : I \longrightarrow S^1$ given by $g(s) = (\cos \pi s, -\sin \pi s)$ lifts to $\tilde{g}(s) = \dfrac{-s}{2}$.

The path $h : I \longrightarrow S^1$ given by $f(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to $\tilde{h}(s) = 2s$.

The first of these examples is shown in Figure 8.

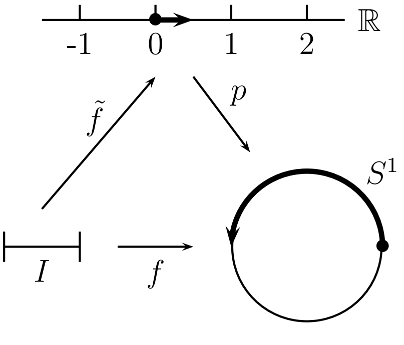


Figure 8

Lemma. ***The Homotopy Lifting Lemma.***
If $p : E \longrightarrow B$ *is a covering map with* $p(e_0) = b_0$*, and* $F : I \times I \longrightarrow B$ *is continuous with* $F(0,0) = b_0$*, then there is a unique lifting of* F *to a continuous map* $\tilde{F} : I \times I \longrightarrow E$ *such that* $\tilde{F}(0,0) = e_0$*. If* F *is a path-homotopy between paths* f_1 *and* f_2*, then their liftings* $\tilde{f}_1$ *and* $\tilde{f}_2$ *end at the same point of* E*, and are path homotopic.*

We can now calculate the fundamental group of the circle. Look at Figure 9. The function f is a loop based at b_0, which is the point $(1,0)$, and $\tilde{f}$ is a lifting of f to the real line. The function p is the covering map $p(x) = (\cos 2\pi x, \sin 2\pi x)$. The point $\tilde{f}(1)$ is in the set $\{z : z = p^{-1}(b_0)\}$, which equals the set of integers. By the Homotopy Lifting Lemma, $\tilde{f}(1)$ depends only on $[f]$, so we can define $\varphi : \pi_1(S^1, b_0) \longrightarrow \mathbb{Z}$ by

$$\varphi([f]) = \tilde{f}(1).$$

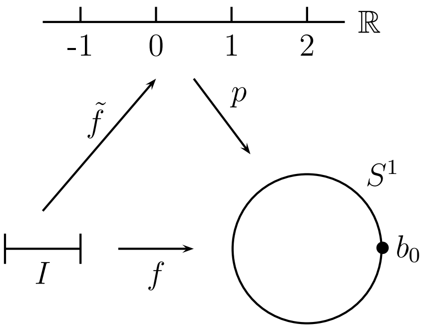


Figure 9

We can show that φ is a homomorphism between $\pi_1(S^1, b_0)$ and $(\mathbb{Z}, +)$:

Let f and g be two loops in S^1 based at b_0, with $\tilde{f}$ and $\tilde{g}$ their liftings to paths in $\mathbb{R}$ beginning at 0. Let $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$. Define $h : I \longrightarrow \mathbb{R}$ by

$$h(s) = \begin{cases} \tilde{f}(2s) & \text{if } s \in [0, \frac{1}{2}] \\ n + \tilde{g}(2s-1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Then h is a path on $\mathbb{R}$ beginning at 0. Now $p(n + x) = p(x)$ for all x, so

$$p(h(s)) = \begin{cases} p(\tilde{f}(2s)) = f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ p(n + \tilde{g}(2s-1)) = p(\tilde{g}(2s-1)) = g(2s-1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Therefore $p \circ h = f * g$, so h is the lifting of $f * g$ beginning at 0. By definition, $\varphi([f * g]) = h(1) = n + m$, so $\varphi([f * g]) = \varphi([f]) + \varphi([g])$. Therefore φ is a homomorphism.

Also φ is bijective:

Say $\varphi([f]) = n = \varphi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g to paths on $\mathbb{R}$ beginning at 0 and ending at n. The real line is simply connected, so $\tilde{f}$ and $\tilde{g}$ are path-homotopic. Let $\tilde{F}$ be the path-homotopy between them. Then the map $F = p \circ \tilde{F}$ is a path homotopy between f and g, and so $[f] = [g]$. Therefore φ is one-one.

For any integer n, there exists a loop f in S^1 based at b_0 defined by $f = p \circ \tilde{f}$ where $\tilde{f} : I \longrightarrow \mathbb{R}$ is a path in $\mathbb{R}$ from 0 to n. By definition, $\varphi([f]) = n$. Therefore φ is onto.

We have now done all the work necessary to calculate the fundamental group of the circle, for φ is a bijective homomorphism, and therefore an isomorphism. This shows that $\pi_1(S^1, b_0)$ is isomorphic to $(\mathbb{Z}, +)$, and so the fundamental group of S^1 is infinite cyclic. Written in group presentation style,

$$\pi_1(S^1, b_0) = \langle a \rangle,$$

where a if the homotopy class of the loop $f : I \longrightarrow S^1$ given by

$$f(s) = (\cos 2\pi s, \sin 2\pi s).$$

The Fundamental Group of a Product Space

Theorem. *The fundamental group $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$ be the projection mappings, and $p_* : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0)$ and $q_* : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(Y, y_0)$ be their induced homomorphisms. From group theory, if $h : C \longrightarrow A$ and $k : C \longrightarrow B$ are group homomorphisms, then $\Phi : C \longrightarrow A \times B$ given by

$$\Phi(c) = (h(c), k(c))$$

is also a group homomorphism, so we can define a homomorphism $\Phi : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by

$$\Phi([f]) = (p_*([f]), q_*([f])) = ([p \circ f], [q \circ f]).$$

We can show that Φ is bijective:

Let $f : I \longrightarrow X \times Y$ be a loop in $X \times Y$ based at (x_0, y_0) such that $p \circ f \simeq_p c_{x_0}$ and $q \circ f \simeq_p c_{y_0}$, where c_{x_0} is the trivial loop in X based at x_0 and c_{y_0} is the trivial loop in Y based at y_0. Let $G : I \times I \longrightarrow X$ and $H : I \times I \longrightarrow Y$ be the respective path-homotopies. Then $F : I \times I \longrightarrow X \times Y$ defined by

$$F(s, t) = (G(s, t), H(s, t))$$

is a path-homotopy between f and the trivial loop in $X \times Y$ based at (x_0, y_0). Therefore the kernel of Φ is trivial, and so Φ is one-one.

Let $g : I \longrightarrow X$ be a loop in X based at x_0 and $h : I \longrightarrow Y$ be a loop in Y based at y_0. Define $f : I \longrightarrow X \times Y$ by $f(s) = (g(s), h(s))$, so f is a loop in $X \times Y$ based at (x_0, y_0). Now $\Phi([f]) = ([p \circ f], [q \circ f]) = ([g], [h])$, and so $([g], [h])$ is in the image of Φ, showing that Φ is onto.

Therefore Φ is an isomorphism between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. □

Example. The Torus.
We can use this method to calculate the fundamental group of the torus $T^2 = S^1 \times S^1$. We know that $\pi_1(S^1, x_0)$ is isomorphic to $(\mathbb{Z}, +)$. Therefore

$$\pi_1(T^2, (x_0, y_0)) \cong \pi_1(S^1, x_0) \times \pi_1(S^1, y_0) \cong \mathbb{Z} \oplus \mathbb{Z},$$

so the fundamental group of the torus is the abelian group with two generators:

$$\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle.$$

We can generalise this argument to show that the fundamental group of the n-dimensional torus $T^n = S^1 \times \ldots \times S^1$ is isomorphic to $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$.

Retraction and the Fundamental Group of the Punctured Plane

Let A be a subspace of a topological space X. We say that A is a **retract** of X if there is a continuous map $r : X \longrightarrow A$ such that $r(a) = a$ for all $a \in A$. We call r a **retraction** of X onto A.

If there is a continuous map $H : X \times I \longrightarrow X$ such that

$$H(x,0) = x \text{ for all } x \in X,$$
$$H(x,1) \in A \text{ for all } x \in X,$$
$$H(a,t) = a \text{ for all } a \in A \text{ and } t \in I,$$

then H is called a **strong deformation retraction**. See Figure 10. At the end of the deformation described by a strong deformation retraction, we have a retraction of X onto A, mapping x into $H(x,1)$.

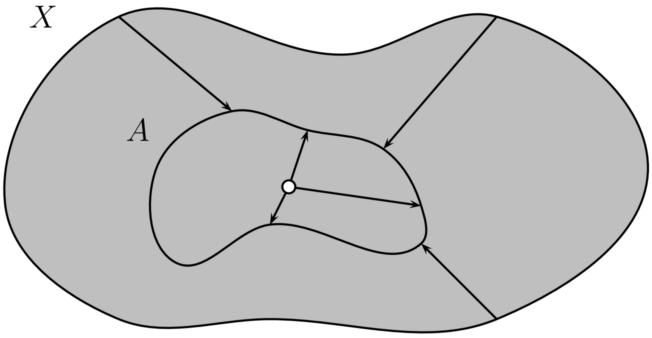


Figure 10

Example. The map $H : (\mathbb{R}^2 - \mathbf{0}) \times I \longrightarrow (\mathbb{R}^2 - \mathbf{0})$ defined by

$$H(x,t) = t\frac{x}{\|x\|} + (1-t)x$$

is a strong deformation retraction of $\mathbb{R}^2 - \mathbf{0}$ onto S^1.

We can show that if A is a strong deformation retract of X, then A is homotopically equivalent to X. Let $j : A \longrightarrow X$ be the inclusion map, let $H : X \times I \longrightarrow X$ be the strong deformation retraction and let $r : X \longrightarrow A$ be defined by $r(x) = H(x,1)$. Then $r \circ j$ equals the identity map of A, and $j \circ r$ is, by the definition of a strong deformation retraction, homotopic to the identity map of X. Therefore j is a homotopy inverse of r. We can see from this that A and X have isomorphic fundamental groups.

Using retraction mappings, it is very easy to calculate the fundamental group of $\mathbb{R}^2 - \mathbf{0}$. We know that there is a strong deformation retraction of $\mathbb{R}^2 - \mathbf{0}$ onto S^1, and we have calculated the fundamental group of S^1: it is infinite cylic. Therefore the fundamental group of the punctured plane is infinite cyclic.

7. SEIFERT AND VAN KAMPEN'S THEOREM

Another method for calculating fundamental groups is this theorem, which H Seifert and E Van Kampen proved independently in the early 1930s. It can be especially useful for finding the fundamental group of a space with the quotient topology.

Theorem. ***Seifert and Van Kampen's Theorem.***
Suppose a topological space X equals $A \cup B$, where A, B and $A \cap B$ are open and $A \cap B$ is path connected. Let $i : A \cap B \longrightarrow A$ and $j : A \cap B \longrightarrow B$ be the inclusion maps, and let $x_0 \in A \cap B$. If

$$\begin{aligned} \pi_1(A, x_0) &= \langle a_1, \ldots, a_n \mid r_1 = \ldots = r_m = 1\rangle, \\ \pi_1(B, x_0) &= \langle b_1, \ldots, b_l \mid s_1 = \ldots = s_k = 1\rangle, \\ \pi_1(A \cap B, x_0) &= \langle c_1, \ldots, c_q \mid t_1 = \ldots = t_p = 1\rangle, \end{aligned}$$

then

$$\pi_1(X, x_0) = \langle a_1, \ldots, a_n, b_1, \ldots, b_l \mid r_1 = \ldots = r_m = s_1 = \ldots = s_k = \\ i_*(c_1)j_*(c_1)^{-1} = \ldots = i_*(c_q)j_*(c_q)^{-1} = 1\rangle.$$

The proof, which is quite long, can be found in [2]. Let us look at some examples.

Example. The Sphere.
Let X be the sphere S^2, let A be the same sphere but punctured at a point x_1, and let B be an open disc on the surface of the sphere about x_1. Then $X = A \cup B$, and A and B are open. The intersect $A \cap B$ equals $B - \{x_1\}$, which is open and path connected. The punctured sphere is homeomorphic to an open disc, and so A and B are both simply connected. The intersect $A \cap B$ is a punctured disc, which is homotopically equivalent to S^1. We now have, for $x_0 \in A \cap B$,

$$\pi_1(A, x_0) = \pi_1(B, x_0) = e,$$

$$\pi_1(A \cap B, x_0) = \langle a\rangle.$$

If $i : A \cap B \longrightarrow A$ and $j : A \cap B \longrightarrow B$ are the inclusion maps, then as both A and B are simply connected, both i_* and j_* are trivial. Therefore, by Seifert and Van Kampen's Theorem, the fundamental group of X is trivial, and so the sphere is simply connected.

Example. The Figure Eight.

Look at Figure 11. The space X is the union of the open spaces A and B, and the intersect $A \cap B$ is open and path connected. The spaces A and B are homotopically equivalent to circles S^1, and $A \cap B$ is a cross, which is homotopically equivalent to a point. Therefore, for $x_0 \in A \cap B$,

$$\pi_1(A, x_0) = \langle a \rangle,$$

$$\pi_1(B, x_0) = \langle b \rangle,$$

$$\pi_1(A \cap B, x_0) = e.$$

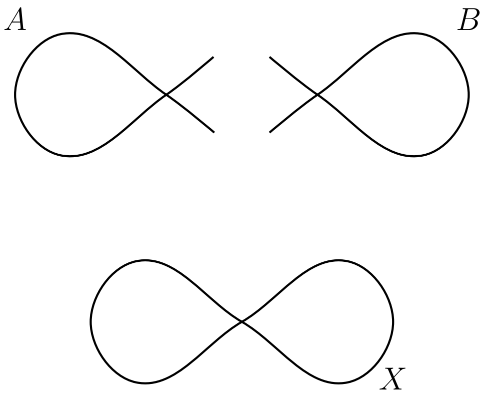


Figure 11

If $i : A \cap B \longrightarrow A$ and $j : A \cap B \longrightarrow B$ are the inclusion maps, then both i_* and j_* are trivial. Therefore, the fundamental group of X is given by

$$\pi_1(X, x_0) = \langle a, b \rangle.$$

We could add another loop C to X meeting at the same point to make a three-leaved clover shape Y. The intersect $X \cap C$ would be trivial again, and so the fundamental group of Y would be the free group on three generators. Similarly, a union of n loops all meeting at the same point has the free group on n generators as its fundamental group.

Example. The Torus.

Let X be the torus T^2 with the quotient topology. Let $A = X - \{x_1\}$ where x_1 is a point, and let B be an open disc about x_1. Then $X = A \cup B$. The intersect $A \cap B$ equals $B - \{x_1\}$, which is path connected, and A, B and $A \cap B$ are all open. All of these spaces are shown in Figure 12. The space A is homotopically equivalent to just the frame of A, which is homeomorphic to a figure eight. The space B is simply connected, and $A \cap B$ is homotopically equivalent to S^1. We now have, for $x_0 \in A \cap B$,

$$\begin{aligned} \pi_1(A, x_0) &= \langle a, b \rangle, \\ \pi_1(B, x_0) &= e \\ \pi_1(A \cap B, x_0) &= \langle c \rangle. \end{aligned}$$

If $i : A \cap B \longrightarrow A$ and $j : A \cap B \longrightarrow B$ are the inclusion maps, then

$$\begin{aligned} i_*(c) &= aba^{-1}b^{-1}, \\ j_*(c) &= 1. \end{aligned}$$

Therefore, the fundamental group of the torus is given by

$$\pi_1(X, x_0) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle.$$

In other words, it is the abelian group with two generators, which is the same answer as when we calculated it using the product topology.

We can calculate the fundamental group of any topolgical space defined with the quotient topology using the same method as above. For example, the fundamental group of the space X in Figure 13 is given by

$$\begin{aligned} \pi_1(X, x_0) &= \langle a, b \mid a^{-1}abb^{-1}ab = 1 \rangle \\ &= \langle a, b \mid ab = 1 \rangle \\ &= \langle a \rangle. \end{aligned}$$

We know that an n-holed torus can be made from a $4n$-sided polygon with side identification labels of the form

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1},$$

and so we can now write down the fundamental groups of all the surfaces Σ_n as defined in the Classification Theorem of Surfaces:

$$\begin{aligned} \pi_1(\Sigma_0, x_0) &= e, \\ \pi_1(\Sigma_1, x_0) &= \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle, \\ \pi_1(\Sigma_n, x_0) &= \langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle \quad \text{for } n > 1. \end{aligned}$$

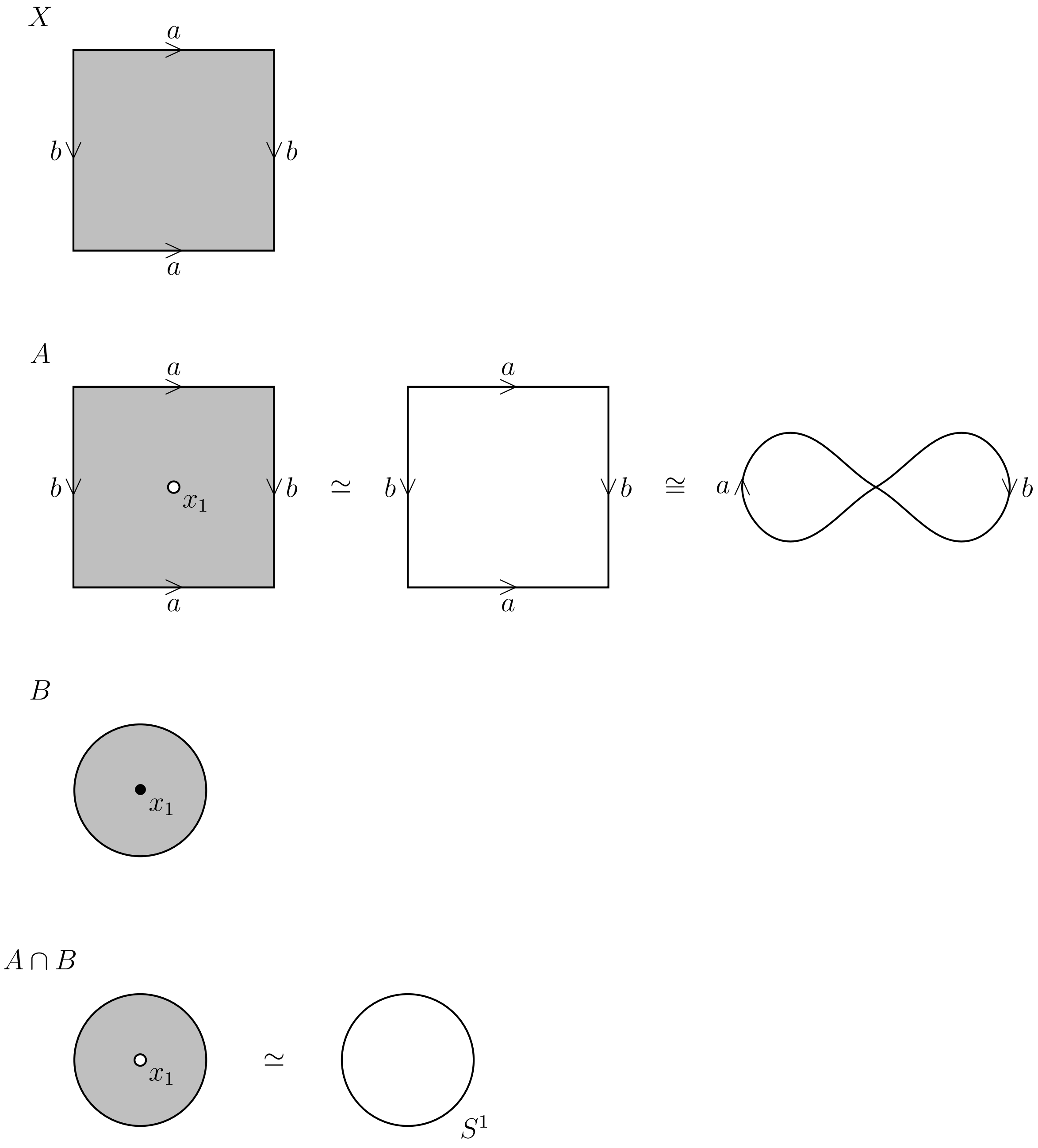


Figure 12

Theorem. *The fundamental groups of the surfaces* Σ_k *and* Σ_l *are not isomorphic for* $k \neq l$.

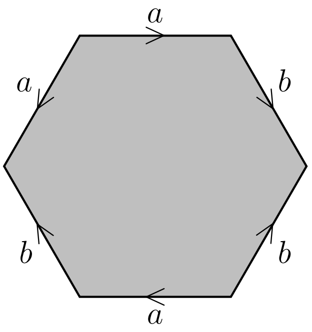


Figure 13

Proof. If G is a group, then G **abelianized** is G with the added relations $xy = yx$ for all $x, y \in G$. We write G abelianized as AG. If we abelianize the fundamental groups of Σ_n we get

$$A\pi_1(\Sigma_0, x_0) = e,$$
$$A\pi_1(\Sigma_1, x_0) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle,$$

$$A\pi_1(\Sigma_n, x_0) = \langle a_1, b_1, \ldots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_na_n^{-1}b_n^{-1} = 1, xyx^{-1}y^{-1} = 1 \rangle \quad \text{for } n > 1,$$

for all $x, y \in a_1, b_1, \ldots, a_n, b_n$. The relation

$$a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_na_n^{-1}b_n^{-1} = 1$$

in the third group is just a consequence of the relation $xyx^{-1}y^{-1} = 1$, and so we can rewrite the group as

$$A\pi_1(\Sigma_n, x_0) = \langle a_1, b_1, \ldots, a_n, b_n \mid xyx^{-1}y^{-1} = 1 \rangle.$$

This shows that the abelianized fundamental group of Σ_n is the abelian group on $2n$ generators, which is isomorphic to $\mathbb{Z}^{2n}$, for all non-negative integer n. The group $\mathbb{Z}^i$ is not isomorphic to the group $\mathbb{Z}^j$ for $i \neq j$, and so the abelianized fundamental groups are not isomorphic. Therefore the fundamental groups of Σ_n are not isomorphic. □

This proves that Σ_k is not homeomorphic to Σ_l for $k \neq l$, which is the part of the Classification Theorem of Surfaces that we said we would prove. Furthermore, we now know that the fundamental group of *any* compact orientable surface Ω_n of genus n is given by:

$$\pi_1(\Omega_0, x_0) = e,$$
$$\pi_1(\Omega_1, x_0) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle,$$
$$\pi_1(\Omega_n, x_0) = \langle a_1, b_1, \ldots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_na_n^{-1}b_n^{-1} = 1 \rangle \quad \text{for } n > 1.$$

8. The Fundamental Theorem of Algebra

We shall finish with an application of the fundamental group. We all know that a polynomial equation of degree n has n complex roots, but proving this is suprisingly difficult when only using algebra. The hardest part is showing that a polynomial has at least one root – the Fundamental Theorem of Algebra – but Algebraic Topology can be used to prove this relatively quickly.

Theorem. *A polynomial equation*

$$x_n + a_{n-1}x^{n-1} \ldots + a_1 x + a_0 = 0 \tag{1}$$

of degree $n > 0$ *(with real or complex coefficients) has at least one (real or complex) root.*

Proof. The theorem will be proved in several stages.

Step 1:

Consider the unit circle on the complex plane. Let $\varphi : I \longrightarrow S^1$ be the loop

$$\varphi(s) = (\cos 2\pi s, \sin 2\pi s) = e^{2\pi i s}.$$

Under the isomorphism of $\pi_1(S^1, b_0)$ with $\mathbb{Z}$, this loop corresponds to the integer 1.

Now consider the map $h : S^1 \longrightarrow S^1$ given by

$$h(z) = z^n.$$

The loop $\psi = h \circ \varphi : I \longrightarrow S^1$ is given by

$$\psi(s) = h(\varphi(s)) = (e^{2\pi i s})^n = e^{2\pi i n s} = (\cos 2\pi n s, \sin 2\pi n s).$$

This lifts to the path $\tilde{\psi}(s) = ns$ in the covering space $\mathbb{R}$, so the loop corresponds to the integer n under the isomorphism of $\pi_1(S^1, b_0)$ with $\mathbb{Z}$. Therefore, the induced homomorphism $h_* : \pi_1(S^1, b_0) \longrightarrow \pi_1(S^1, b_0)$ takes a generator of the infinite cyclic group to n times itself.

Step 2:

We will now show that if a function $k : S^1 \longrightarrow Y$ is homotopic to a constant map (we say k is **inessential**), then k_* is the trivial homomorphism.

Let $P : S^1 \times I \longrightarrow B^2$, where B^2 is the closed unit ball on $\mathbb{R}^2$, be defined by

$$P(x,t) = (1-t)x.$$

Then P takes $S^1 \times [0,1)$ bijectively onto $B^2 - \mathbf{0}$, and maps $S^1 \times \{1\}$ to the point $\mathbf{0}$. Suppose $K : S^1 \times I \longrightarrow Y$ is a homotopy between k and the constant map c which takes the whole of S^1 to $y_0 \in Y$. As K is constant on the set $S^1 \times \{1\}$, we can define a map $g : B^2 \longrightarrow Y$ such that $g \circ P = K$, as shown in Figure 14. The map g is an extension of k, for if $x \in S^1$, then

$$g(x) = g(P(x,0)) = K(x,0) = k(x).$$

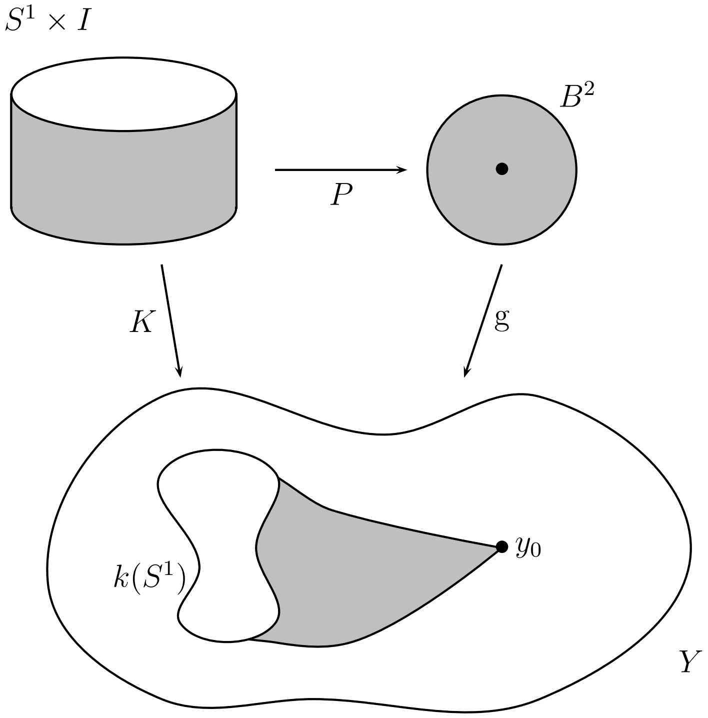


FIGURE 14

Now let $j : S^1 \longrightarrow B^2$ be the inclusion map. Then $g \circ j = k$. By the functorial properties of the induced homomorphisms, $k_* = g_* \circ j_*$. But the range of j_* is the fundamental group of B^2, which is the trivial group since B^2 is a convex subset of $\mathbb{R}^2$. Therefore, j_* is the trivial homomorphism, and so k_* is the trivial homomorphism.

Step 3:

We will now show that the equation

$$z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$$

with

$$|a_{n-1}| + \ldots + |a_1| + |a_0| < 1$$

has a root lying in the closed unit ball B^2.

Assume for a contradiction that the equation has no root in B^2. Then we can define a map $g : B^2 \longrightarrow \mathbb{R}^2 - \mathbf{0}$ by

$$g(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0.$$

Let $f : S^1 \longrightarrow \mathbb{R}^2 - \mathbf{0}$ be the restriction of g to S^1, and define $F_1 : S^1 \times I \longrightarrow \mathbb{R}^2 - \mathbf{0}$ by

$$F_1(z, t) = g(tz).$$

Then F_1 is a homotopy between a constant map and f. Since by our assumption $g(z) \neq 0$ for all $z \in B^2$, F_1 never vanishes.

Now define $F_2 : S^1 \times I \longrightarrow \mathbb{R}^2 - \mathbf{0}$ by

$$F_2(z, t) = z^n + (t)(a_{n-1}z^{n-1} + \ldots + a_1 z + a_0).$$

This is a homotopy between the map $k : S_1 \longrightarrow \mathbb{R}^2 - \mathbf{0}$ defined by

$$k(z) = z^n$$

and f, and it never vanishes because

$$\begin{aligned}
|F_2(z,t)| &\geqslant |z^n| - \left|t(a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)\right| \\
&\geqslant 1 - t(\left|a_{n-1}z^{n-1}\right| + \ldots + |a_1 z| + |a_0|) \\
&= 1 - t(|a_{n-1}| + \ldots + |a_1| + |a_0|) \\
&> 0.
\end{aligned}$$

Now if $h : S^1 \longrightarrow S^1$ is given by $h(z) = z^n$ as in Step 1, and $j : S^1 \longrightarrow \mathbb{R}^2 - \mathbf{0}$ is the inclusion map, then $k = j \circ h$, and $k_* = h_* \circ h_*$ by the functorial properties. Since h_* is "multiplication by n" and j_* is an isomorphism, k_* is *not* the trivial homomorphism, and so k is not homotopic to a constant map, by Step 2. However, this contradicts the fact that f, which is homotopic to k, *is* homotopic to a constant map. Therefore the equation has a root in B^2.

Step 4:

Given Equation (1),

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0,$$

substitute $x = cy$, where c is a real positive number. We get

$$(cy)^n + a_{n-1}(cy)^{n-1} + \ldots + a_1(cy) + a_0 = 0.$$

Dividing through by c^n gives us

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + \ldots + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0. \tag{2}$$

If we choose c large enough, we can ensure that

$$\left|\frac{a_{n-1}}{c}\right| + \ldots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1.$$

So, by Step 3, Equation (2) has at least one root, say $y = y_0$. But if y_0 is a root of Equation (2), then $x_0 = cy_0$ is a root of Equation (1). Therefore our original equation has at least one root. □

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